Week 10

Nov 8 —

Lecture

(I) More on Curves
(II) Conservative Vector Fields
(III) Independence of Path
(IV) The Component Test

(I) More on Curves.

We introduced two kinds of line integrals

$$\int_C f\,ds \quad \text{and} \quad \int_C \vec{F}\cdot d\vec{r}.$$

The first one is indpt of the parametrization and direction of the curve. The second one is indpt of parametrization of the same orientation but changes sign when the orientation is reversed.

Let C be an oriented curve that admits a parametrization

$$\vec{r}:[a,b]\to\mathbb{R}^2,\mathbb{R}^3,\quad \vec{r}(a)=\vec{P},\ \vec{r}(b)=\vec{Q}.$$

The parametrization $\vec{r}_1:[a,b]\to\mathbb{R}^2,\mathbb{R}^3$, $\vec{r}_1(t)=\vec{r}(a+b-t)$, describe the same curve but with the reverse orientation, $\vec{r}_1(a)=\vec{Q}$ and $\vec{r}_1(b)=\vec{P}$, and

$$\vec{r}_1{}'(t)=-\vec{r}\,'(a+b-t)$$

Therefore, if we let $-C$ denote this new oriented curve

$$\int_{-C}\vec{F}\cdot d\vec{r}=\int_a^b \vec{F}(\vec{r}_1(t))\cdot\vec{r}_1{}'(t)\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(a+b-t)) \cdot \vec{r}_1'(t)\, dt$$

$$= -\int_b^a \vec{F}(\vec{r}(z)) (-\vec{r}'(z))\, dz \quad , \qquad z = a+b-t, \ dz = -dt$$

$$= -\int_a^b \vec{F}(\vec{r}(z)) \cdot \vec{r}'(z)\, dz$$

$$= -\int_C \vec{F} \cdot d\vec{r} .$$

We conclude

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r} .$$

Next, let C_1 and C_2 be two oriented curves on $[a, b]$ and $[c, d]$ respectively. When $\vec{r}_1(b) = \vec{r}_2(c)$, we can put them together to form a piecewise curve and denote it by $C_1 + C_2$.

Similarly, one can define $C_1 + C_2 + \cdots + C_n$.

II Conservative Vector fields

Let $\vec{F}$ be a smooth vector field defined in an open region. It is called conservative or gradient if there exists a function g s.t.

$$\vec{F} = \nabla g, \text{ ie } \frac{\partial g}{\partial x} = M, \ \frac{\partial g}{\partial y} = N, \ \frac{\partial g}{\partial z} = P \quad (n=3)$$

$$\text{ie } \frac{\partial g}{\partial x} = M, \ \frac{\partial g}{\partial y} = N, \quad (n=2).$$

The function g is the potential of $\vec{F}$.

Conservative v.f's are important because of their role in physics.

<u>Theorem 1</u> Let C be an oriented curve running from $\vec{P}$ to $\vec{Q}$ in the open region G on which a conservative v.f. $\vec{F}$ is defined. Then

$$\int_C \vec{F}\cdot d\vec{r} = g(\vec{Q}) - g(\vec{P}).$$

In particular, when C is a closed curve,

$$\oint_C \vec{F}\cdot d\vec{r} = 0.$$

Pf: Since g is the potential of $\vec{F}$, $\nabla g = \vec{F}$, so by the chain rule,

$$\frac{d}{dt} g(\vec{r}(t)) = \frac{d}{dt} g(x(t), y(t), z(t))$$
$$= \frac{\partial g}{\partial x}(\vec{r}(t))\, x'(t) + \frac{\partial g}{\partial y}(\vec{r}(t))\, y'(t) + \frac{\partial g}{\partial z}(\vec{r}(t))\, z'(t).$$

$\therefore$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt$$
$$= \int_a^b \left[M(\vec{r}(t))\, x'(t) + N(\vec{r}(t))\, y'(t) + P(\vec{r}(t))\, z'(t)\right] dt$$
$$= \int_a^b \frac{\partial g}{\partial x}(\vec{r}(t))\, x'(t) + \frac{\partial g}{\partial y}(\vec{r}(t))\, y'(t) + \frac{\partial g}{\partial z}(\vec{r}(t))\, z'(t)$$
$$= \int_a^b \frac{d}{dt} g(\vec{r}(t))\, dt$$

$= g(\vec{r}(b)) - g(\vec{r}(a))$

$= g(\vec{Q}) - g(\vec{P})$. #

e.g. 1. Find the work done by the conservative v.f.

$$\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

when $g = xyz$ from $(-1,3,9)$ to $(1,6,-4)$.

Using thm 1, the work done

$$\int_C \vec{F}\cdot d\vec{r}$$

from $\vec{P}(-1,3,9)$ to $\vec{Q}(1,6,-4)$ is indpt of the choice of C

$$\therefore \int_C \vec{F}\cdot d\vec{r} = g(\vec{Q}) - g(\vec{P})$$

$$= g((1,6,-4)) - g((-1,3,9))$$

$$= -24 + 27$$

$$= 3 \quad \#$$

Note. When $\vec{F}$ is conservative, we write

$$\int_{(-1,3,9)}^{(1,6,-4)} \vec{F}\cdot d\vec{r}$$

since no matter which curve you run from $(-1,3,9)$ to $(1,6,-4)$, the work done is the same!

(III) Independence of Path.

A v.f. $\vec{F}$ is indept of path if for any two pts $\vec{P}, \vec{Q}$ in G and any curve running from $\vec{P}$ to $\vec{Q}$, the work done is the same.

<u>Theorem 2</u> A v.f. is independent of path if and only if it is conservative.

Pf $\Leftarrow$) From Thm 1,

$$\int_C \vec{F}\cdot d\vec{r} = g(\vec{Q})-g(\vec{P}).$$

Hence if C_1, C_2 also run from $\vec{P}$ to $\vec{Q}$,

$$\int_{C_1} \vec{F}\cdot d\vec{r} = g(\vec{Q})-g(\vec{P})$$

$$= \int_{C_2} \vec{F}\cdot d\vec{r}.$$

$\Rightarrow$) (take $n=2$. The general n-case the same)

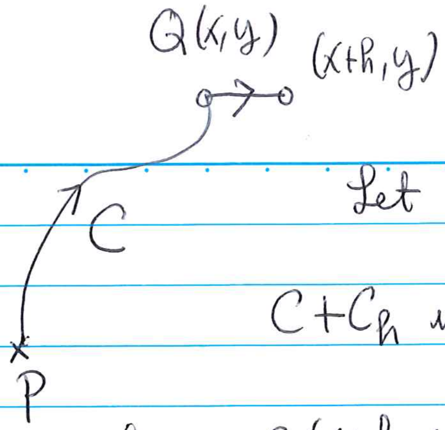
Let $P(x_0, y_0)$ and $Q(x, y)$ be in the open region G.

Define

$$g(x,y) = \int_C \vec{F}\cdot d\vec{r}$$

where C is an oriented curve from $\vec{P}$ to $\vec{Q}$. g is well-defined by assumption. We claim

$$\frac{\partial g}{\partial x}(x,y) = M(x,y) \quad \text{when } \vec{F} = M\hat{\imath} + N\hat{\jmath}.$$



Let $C_h : \vec{r}(t) = (x + th, y),\quad t \in [0,1]$

$C + C_h$ is an oriented curve from P to $(x+th, y)$

then $$g(x+h, y) = \int_{C+C_h} \vec{F} \cdot d\vec{r}$$

$$\therefore\quad g(x+h,y) - g(x,y) = \int_{C+C_h} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F} \cdot d\vec{r} + \int_{C_h} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{C_h} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \vec{F}(x+th, y) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \vec{F}(x+th, y) \cdot (h\hat{i} + 0\hat{j})\, dt$$

$$= h \int_0^1 M(x+th, y)\, dt.$$

$$\therefore\quad \frac{\partial g}{\partial x}(x,y) = \lim_{h \to 0} \frac{g(x+h,y) - g(x,y)}{h}$$

$$= \lim_{h \to 0} \int_0^1 M(x+th, y)\, dt$$

$$= M(x,y).$$

Similarly, one shows $\frac{\partial g}{\partial y} = N$. #

(IV) The component Test

Theorem 3 Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$. When $\vec{F}$ is conservative,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \quad \text{hold.}$$

When $\vec{F} = M\hat{i} + N\hat{j}$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ holds.

Pf: $\vec{F} = \nabla g$ as assumed. Then

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\frac{\partial g}{\partial y} = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = \frac{\partial}{\partial z}\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\frac{\partial g}{\partial z} = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = \frac{\partial}{\partial z}\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\frac{\partial g}{\partial z} = \frac{\partial P}{\partial y}. \quad \#$$

e.g. 3 Show that $\vec{F} = xy\hat{i} + z\hat{j} + \cos x\,\hat{k}$ is not conservative.

$M = xy$, $N = z$, $P = \cos x$.

Now, $\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 0$.

By Component Test, $\vec{F}$ has no potential.

e.g. 4 Find a potential for

$$(e^x \cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy+z)\hat{k}.$$

First, check Component Test:

$$M = e^x \cos y + yz, \quad N = xz - e^x \sin y, \quad P = xy + z$$

$$\frac{\partial M}{\partial y} = -e^x \sin y + z, \quad \frac{\partial N}{\partial x} = z - e^x \sin y, \quad \text{yes.}$$

$$\frac{\partial M}{\partial z} = y, \quad \frac{\partial P}{\partial x} = y, \quad \text{yes.}$$

$$\frac{\partial N}{\partial z} = x, \quad \frac{\partial P}{\partial y} = x, \quad \text{yes.}$$

the Test passes.

$$\frac{\partial g}{\partial x} = e^x \cos y + yz$$

$$\Rightarrow g = e^x \cos y + xyz + h(y,z) \text{ for some } h,$$

$$\frac{\partial g}{\partial y} = -e^x \sin y + xz + \frac{\partial h}{\partial y}(y,z)$$

$$= N = xz - e^x \sin y$$

$$\Rightarrow \frac{\partial h}{\partial y}(y,z) = 0, \text{ ie, } h(y,z) = j(z) \text{ for some } j.$$

$$\frac{\partial g}{\partial z} = xy + \frac{\partial j}{\partial z}(z) = xy + z$$

$$\Rightarrow \frac{\partial j}{\partial z}(z) = z.$$

$$\Rightarrow j(z) = \frac{1}{2}z^2 + C, \quad C \text{ any constant.}$$

$g(x,y,z) = e^x \cos y + xyz + \frac{1}{2} z^2 + C.$ #

e.g. 5 ** (Important).

Show that $\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j} + 0\hat{k}$ has no potential.

$\vec{F}$ is defined in the open region

$$\{(x,y,z) : x^2+y^2 \neq 0\} \subset \mathbb{R}^3$$

Let $C : \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}$, $t \in [0, 2\pi]$,
be the circle on the xy-plane.

$$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 + \sin^2 t}\hat{i} + \frac{\cos t}{\cos^2 t + \sin^2 t}\hat{j} \right) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j})\, dt$$

$$= \int_0^{2\pi} (\sin t)(-\sin t) + \cos t \cos t \quad dt$$

$$= 2\pi \neq 0$$

By theorem 1, $\vec{F}$ has no potential.

But, interestingly, $\vec{F}$ satisfies the Component Test:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x} \quad !$$

$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} = 0, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} = 0 \,!$$

Finally, the expression

$$M\,dx + N\,dy + P\,dz$$

is called a differential form. A differential form is exact if $\exists\, g$ s.t.

$$M\hat{i} + N\hat{j} + P\hat{k} = \nabla g, \quad \text{ie,}$$

$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ has a potential.

e.g. Show that $y\,dx + x\,dy + 4\,dz$ is exact and find

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz.$$

$$\frac{\partial g}{\partial x} = y \Rightarrow g = xy + h(y,z)$$

$$\frac{\partial g}{\partial y} = x \Rightarrow x + \frac{\partial h}{\partial y}(y,z) = x$$

$$\Rightarrow h(y,z) = j(z)$$

$$\frac{\partial g}{\partial z} = 4 \Rightarrow j'(z) = 4$$

$$\Rightarrow j(z) = 4z + C$$

$\therefore\ g(x,y,z) = xy + 4z + C$ is the potential.

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = g((2,3,-1)) - g((1,1,1))$$

$$= -3 \ \#$$